

Theory of Fermion to Boson Mappings: Old Wine in New Bottles

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Abstract

After a brief review of various mappings of fermion pairs to bosons, we rigorously derive a general approach. Following the methods of Marumori and Otsuka, Arima, and Iachello, our approach begins with mapping states and constructs boson representations that preserve fermion matrix elements. In several cases these representations factor into finite, Hermitian boson images times a projection or norm operator that embodies the Pauli principle. We pay particular attention to truncated boson spaces, and describe general methods for constructing Hermitian and approximately finite boson image Hamiltonians, including effective operator theory to account for excluded states. This method is akin to that of Otsuka, Arima, and Iachello introduced in connection with the Interacting Boson Model, but is more rigorous, general, and systematic.

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I. INTRODUCTION

The original attempt at bosonization of the nuclear many-fermion system was motivated by collective particle-hole modes in nuclei [1,2]. Since that time the interacting boson model [3] (IBM) has been phenomenologically very successful in explaining low energy nuclear spectroscopy for heavy nuclei. The bosons in this model are thought to represent monopole ($J=0$), quadrupole ($J=2$), and sometimes hexadecapole ($J=4$) correlated pairs of valence nucleons in the shell model. The IBM Hamiltonian is Hermitian, usually has at most two-boson interactions, and conserves boson number, reflecting the particle-particle, rather than particle-hole, nature of the underlying fermion pairs. While one can numerically diagonalize the general IBM Hamiltonian, one of the strengths of IBM is the existence of algebraic limits corresponding to the subgroups $SU(3)$, $U(5)$, or $O(6)$, with analytic expressions for excitation bands and transition strengths, which encompass an enormous amount of nuclear data.

The microscopic reasons for the success of such a simple model are elusive. Otsuka, Arima, and Iachello, along with Talmi, have used a mapping of the shell model Hamiltonian to the IBM Hamiltonian [4,5] based on the seniority model [6], but these attempts have not done well for well-deformed nuclei [7]. For this reason we have revisited boson mappings to see if we can understand the success of the IBM starting from the shell model.

We will begin by sketching out various historic approaches to boson mappings [8,9]. We then specifically follow Marumori [2] and Otsuka et al. [4,5] (OAI) in our mapping procedure which maps fermion states into boson states and construct boson operators that reproduce fermion matrix elements. We give the boson representation of the Hamiltonian and review the result that, in the full Boson Fock space, it factorizes into a boson image, which is the same as the Belyaev-Zelevinskii Hamiltonian [1]—in fact in the full space all mappings yield the same results—times a normalization operator which projects out the spurious states. However, since our goal is to understand the IBM, which only deals with a few of the enormous degrees of freedom of the shell model, we go on to discuss boson images

in truncated spaces. This, we shall see, gives rigorous insight into the OAI mapping and shows how to systematically extend it.

II. A BRIEF HISTORY OF BOSON MAPPINGS

The fundamental goal is to solve the many-fermion Schrödinger equation

$$\hat{H} |\Psi_\lambda\rangle = E_\lambda |\Psi_\lambda\rangle \quad (1)$$

and find transition matrix elements between eigenstates, $t_{\lambda\lambda'} = \langle \Psi_\lambda | \hat{T} | \Psi_{\lambda'} \rangle$. For fermion many-body (shell-model) basis states one often uses Slater determinants, antisymmetrized products of single-fermion wavefunctions which we can write using Fock creation operators: $a_{i_1}^\dagger \cdots a_{i_n}^\dagger |0\rangle$ for n fermions. For an even number of fermions one can instead construct states from products of fermion pairs,

$$|\Psi_\beta\rangle = \prod_{m=1}^N \hat{A}_{\beta_m}^\dagger |0\rangle; \quad (2)$$

if the number of fermion is fixed at n then m runs from 1 to $N = n/2$. As the fermion Fock space may be so large as to make direct solution intractable, the idea of a boson mapping is to replace the fermion operators with boson operators, *using only a minimal number of boson degrees of freedom*, that approximate the spectrum and transition matrix elements of the original fermion problem. There are two approaches to boson mappings which we now review.

The first approach, epitomized in nuclear physics by Belyaev and Zelevinskii (BZ) [1], is to map fermion operators to boson operators so as to preserve the original algebra. Specifically, consider a space with 2Ω single-fermion states; a_i^\dagger, a_j signify fermion creation and annihilation operators. The set of all bilinear fermion operators, $a_i a_j, a_k^\dagger a_l^\dagger, a_i^\dagger a_j$, form the Lie algebra of $SO(4\Omega)$, as embodied by the commutation relations

$$[a_i a_j, a_k a_l] = 0 \quad (3)$$

$$[a_i a_j, a_k^\dagger a_l^\dagger] = \delta_{il} \delta_{jk} + \delta_{ik} a_l^\dagger a_j + \delta_{jl} a_k^\dagger a_i - (i \leftrightarrow j) \quad (4)$$

$$[a_i a_j, a_k^\dagger a_l] = \delta_{jk} a_i a_l - (i \leftrightarrow j) \quad (5)$$

$$[a_i^\dagger a_j, a_k^\dagger a_l] = \delta_{jk} a_i^\dagger a_l - \delta_{il} a_k^\dagger a_j \quad (6)$$

At this point it is convenient to introduce collective fermion pair operators

$$\hat{A}_\beta^\dagger \equiv \frac{1}{\sqrt{2}} \sum_{ij} (\mathbf{A}_\beta^\dagger)_{ij} a_i^\dagger a_j^\dagger. \quad (7)$$

We always choose the $\Omega(2\Omega - 1)$ matrices \mathbf{A}_β to be antisymmetric so as to preserve the underlying fermion statistics, thus eliminating the need later on to distinguish between ‘ideal’ and ‘physical’ bosons. We also assume the following normalization and completeness relations for the matrices:

$$\text{tr } \mathbf{A}_\alpha \mathbf{A}_\beta^\dagger = \delta_{\alpha\beta}; \quad (8)$$

$$\sum_\alpha (\mathbf{A}_\alpha^\dagger)_{ij} (\mathbf{A}_\alpha)_{j'i'} = \frac{1}{2} (\delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{ji'}). \quad (9)$$

Generic one- and two-body fermion operators we represent by $\hat{T} \equiv \sum_{ij} T_{ij} a_i^\dagger a_j$, $\hat{V} \equiv \sum_{\mu\nu} \langle \mu | V | \nu \rangle \hat{A}_\mu^\dagger \hat{A}_\nu$, where $T_{ij} = \langle i | \hat{T} | j \rangle$; from such operators one can construct a fermion Hamiltonian \hat{H} . Now one has the following commutation relations:

$$[\hat{A}_\alpha, \hat{A}_\beta] = [\hat{A}_\alpha^\dagger, \hat{A}_\beta^\dagger] = 0; \quad (10)$$

$$[\hat{A}_\alpha, \hat{A}_\beta^\dagger] = \delta_{\alpha\beta} - 2 \sum_{ij} (\mathbf{A}_\beta^\dagger \mathbf{A}_\alpha)_{ij} a_i^\dagger a_j; \quad (11)$$

$$[\hat{A}_\alpha, \hat{T}] = 2 \sum_\beta \text{tr } (\mathbf{A}_\alpha \mathbf{T} \mathbf{A}_\beta^\dagger) \hat{A}_\beta \quad (12)$$

$$[\hat{T}_1, \hat{T}_2] = \sum_{ij} (\mathbf{T}_1 \mathbf{T}_2 - \mathbf{T}_2 \mathbf{T}_1)_{ij} a_i^\dagger a_j \quad (13)$$

The method of Belyaev and Zelevinskii is to find boson images of the bifermion operators,

$$(\hat{A}_\mu^\dagger)_B = b_\mu^\dagger + \sum_{\alpha\beta\gamma} x_\mu^{\alpha\beta\gamma} b_\alpha^\dagger b_\beta^\dagger b_\gamma + \sum_{\alpha\beta\gamma\delta\epsilon} x_\mu^{\alpha\beta\gamma\delta\epsilon} b_\alpha^\dagger b_\beta^\dagger b_\gamma^\dagger b_\delta b_\epsilon + \dots \quad (14)$$

$$(\hat{A}_\mu)_B = (A_\mu^\dagger)_B^\dagger \quad (15)$$

$$(\hat{T})_B = \sum_{\alpha\beta} y^{\alpha\beta} b_\alpha^\dagger b_\beta + \sum_{\alpha\beta\gamma\delta} y^{\alpha\beta\gamma\delta} b_\alpha^\dagger b_\gamma^\dagger b_\beta b_\delta + \dots \quad (16)$$

where $b_\alpha, b_\beta^\dagger$ are boson creation and annihilation operators, $[b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}$, with the coefficients x, y chosen so that the images $(A_\mu^\dagger)_B, (A_\nu)_B, (T)_B$ have the same commutation relations as

in (10)-(13). Because the algebra is exactly matched, if one builds boson states in exact analogy to the fermion states then the full boson Fock space is not spanned and one does not have nonphysical or spurious states.

In the full boson Fock space, that is, no truncation of the boson degrees of freedom, the image of one body operators is finite and given quite simply by $(\hat{T})_B = 2 \sum_{\alpha\beta} \text{tr}(\mathbf{A}_\alpha \mathbf{T} \mathbf{A}_\beta^\dagger) b_\alpha^\dagger b_\beta$. Since in the full space any fermion Hamiltonian can be written in terms of one-body operators, the boson image of a finite fermion Hamiltonian will be finite in the full space. The states that one must use then are built from the boson representations of the fermion pairs given in (14) which will not just be products of bosons but will include exchange terms. For example, for two bosons and using (14),

$$\hat{A}_\alpha^\dagger \hat{A}_\beta^\dagger |0\rangle \rightarrow (b_\alpha^\dagger b_\beta^\dagger + x_\alpha^{\sigma\tau\beta} b_\sigma^\dagger b_\tau^\dagger) |0\rangle. \quad (17)$$

These exchange terms are due to the antisymmetry. We shall take care of such exchange effects by introducing a norm operator in the boson space.

For *truncated spaces*, however, blind application of the boson representations of the pair operators (14) will produce states outside the truncated space. Therefore, if these states are omitted, then the Hamiltonian needs to be renormalized to account for these omissions, which in general leads to an infinite BZ Hamiltonian. Marshalek [10] points out that there exist mappings that are both finite and Hermitian, but these in general require projection operators to eliminate spurious states. We will regain this result later on in this paper.

A variant of Belyaev-Zelevinskii that also preserves commutation relations is the Dyson mapping [11]:

$$\hat{A}_\alpha \rightarrow b_\alpha; \quad (18)$$

$$\hat{A}_\beta^\dagger \rightarrow b_\beta^\dagger - 2 \sum_{\lambda\mu\nu} \text{tr}(\mathbf{A}_\lambda \mathbf{A}_\beta^\dagger \mathbf{A}_\mu \mathbf{A}_\nu^\dagger) b_\lambda^\dagger b_\mu^\dagger b_\nu \quad (19)$$

$$\hat{T} \rightarrow 2 \text{tr} \sum_{\alpha\beta} (\mathbf{A}_\alpha \mathbf{T} \mathbf{A}_\beta^\dagger) b_\alpha^\dagger b_\beta. \quad (20)$$

The operators are then clearly finite; on the other hand they are just as clearly non-Hermitian. From a computational viewpoint non-Hermiticity is only a minor barrier, but

it is an obstacle to an understanding of the microscopic origin of Hermitian IBM Hamiltonians. Furthermore, unless the truncated pairs constitute a subalgebra, under truncation, renormalization will produce an infinite expansion, although it may be possible that this expansion may be more convergent than the BZ expansion.

The second major approach, pioneered by Marumori [2], is to map fermion states and construct boson representation operators that preserve matrix elements. The original work of Marumori, however, focused on particle-hole excitations and so the number of pairs and consequently bosons were not fixed. However, this method can be applied to particle-particle pairs as well.

Marumori constructs the norm matrix

$$\mathcal{N}_{\alpha\beta} = \langle \Psi_\alpha | \Psi_\beta \rangle \quad (21)$$

and then the Usui operator

$$U = \sum_{\alpha,\beta;n} |\Phi_\beta\rangle (\mathcal{N})_{\beta\alpha}^{-1/2} \langle \Psi_\alpha| \quad (22)$$

where bosons states are constructed in strict analogy to the fermion states,

$$|\Phi\rangle = \prod_{m=1}^N b_{\beta_m}^\dagger |0\rangle. \quad (23)$$

Then the Marumori expansion of any fermion operator is

$$O_B = U O_F U^\dagger. \quad (24)$$

We shall show that, in the full space, these boson representation operators factor into a finite boson image times a norm operator. Furthermore, the boson image of the Hamiltonian is the same as the BZ image, and hence, the two methods are equivalent in the full space.

Otsuka, Arima, and Iachello (OAI), along with Talmi [4,5], investigated the microscopic origins of the Interacting Boson Model through boson mappings. Although they also mapped states, they differed from Marumori in some key details. First of all, they built states built on a fixed number of particle-particle, not particle-hole, pairs. In addition, the space was

truncated to include only one monopole ($J^\pi = 0^+$) and quadrupole ($J^\pi = 2^+$) pair. These states were orthogonalized based on seniority. That is, they constructed, for $2N$ fermions, low-seniority basis states of S and D fermion pairs, $|S^{N-n_d}D^{n_d}\rangle$, and then orthonormalized the states such that the zero-seniority state is mapped to itself, and states of higher seniority v were orthogonalized against states of lower seniority,

$$|v\rangle \rightarrow |"v"\rangle = |v\rangle + |v-2\rangle + |v-4\rangle + \dots \quad (25)$$

Then OAI calculated the matrix elements $\langle "S^{N-n'_d}D^{n'_d}" | H_F | "S^{N-n_d}D^{n_d}" \rangle$ for $n_d, n'_d = 0, 1, 2$ and obtained the coefficients for their one plus two-boson Hamiltonian. These coefficients have an implicit N -dependence (and for large N and arbitrary systems such matrix elements are not trivial to calculate, especially in analytic form!) and thus a many-body dependence. At first sight this is not entirely unreasonable as it is well known the IBM parameters change substantially as a function of the number of bosons, even within a major ‘shell’. Nonetheless the OAI mapping has three drawbacks. The first is that it is not clear how to systematically calculate many-body contributions beyond that contained in the OAI prescription, whereas the method we shall describe is fully and rigorously systematic. The second is that the OAI prescription can induce many-body effects where none are needed. This point will be illustrated in section V C. Thirdly, only the $n_d = 0, 1, 2$ space is exactly mapped, but very deformed systems will involve large n_d . In fact, for an axial rotor limit, the average number of d-bosons in the ground state band is $2/3$ the total number of bosons. Correcting this by also mapping matrix elements with $n_d > 2$ will involve many-body terms.

As an alternative to OAI, Skouras, van Isacker, and Nagarajan [12] proposed a “democratic” mapping where the orthogonalization is based on eigenvectors of the norm matrix rather than seniority.

In what follows we present a unified state-mapping method and obtain four strong results. First we derive matrix elements of the fermion operators in the pair basis (2). Second, we give general expressions for fermion matrix elements via boson representations. Third, we show how in several cases one can have exact, finite, and Hermitian boson images of fermion

operators. Finally, we show how to extend the OAI and democratic mappings in a systematic and rigorous fashion, and illustrate how the choice of orthogonalization can affect the many-body dependence of the boson images.

III. MATRIX ELEMENTS OF FERMION-PAIR STATES

The starting point of any state-mapping method is the calculation of matrix elements of fermion operators between states constructed from fermion pairs of the form (2), including the overlap: $\langle \Psi_\alpha | \Psi_\beta \rangle$, $\langle \Psi_\alpha | \hat{H} | \Psi_\beta \rangle$, $\langle \Psi_\alpha | \hat{T} | \Psi_\beta \rangle$, and so on. These matrix elements are much more difficult to compute than the corresponding matrix elements between Slater determinants. As we shall show, however, full and careful attention paid to the problem of calculation matrix elements can yield powerful results. Silvestre-Brac and Piepenbring [13], laboriously using commutation relations, derived a Wick theorem for fermion pairs. Rowe, Song and Chen [14] using ‘vector coherent states’ (we would say fermion-pair coherent states) found matrix elements between pair-condensate wavefunctions, states of the form $(\hat{A}^\dagger)^N |0\rangle$. Using a theorem by Lang et al. [15], we have generalized the method of Rowe, Song and Chen and recovered (actually discovered independently) the expressions of Silvestre-Brac and Piepenbring.

To derive the generalized Wick’s theorem, we use the following theorem (its proof, which requires fermion coherent states and integration over Grassmann variables, is found in Appendix A of [15]):

Let \hat{U} be an operator of the form

$$\hat{U} \equiv \exp(\hat{h}(n)) \cdots \exp(\hat{h}(2)) \exp(\hat{h}(1)), \quad (26)$$

where each $\hat{h}(t)$ is of the form

$$\hat{h}(t) = \sum_{ij} [T(t)_{ij} \hat{a}_i^\dagger \hat{a}_j + B(t)_{ij} \hat{a}_i \hat{a}_j + A^*(t)_{ji} \hat{a}_i^\dagger \hat{a}_j^\dagger]. \quad (27)$$

Again, the matrices \mathbf{A} , \mathbf{B} are antisymmetric (\mathbf{T} is not, in general), and are of dimension

$2\Omega \times 2\Omega$. We introduce the $4\Omega \times 4\Omega$ matrix representation of \hat{U} in the basis of the single-particle Fock operators a_i^\dagger, a_j . Then:

$$\begin{pmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{pmatrix} = \prod_{t=1}^n \exp \begin{pmatrix} \mathbf{T}(t) & 2\mathbf{A}^\dagger(t) \\ 2\mathbf{B}(t) & -\mathbf{T}^T(t) \end{pmatrix}. \quad (28)$$

The conclusion of the theorem is the vacuum expectation value of \hat{U} :

$$\langle 0 | \hat{U} | 0 \rangle = \sqrt{\det(\mathbf{U}_{22})} \exp \left(\frac{1}{2} \sum_t \text{tr} \mathbf{T}(t) \right). \quad (29)$$

Appropriate derivatives of (29) bring down pair creation and annihilation operators, leading to the desired matrix elements. We do this in some detail for the overlap matrix elements $\langle \Psi_\alpha | \Psi_\beta \rangle$ for the states given in (2); then we simply give the results for matrix elements of one- and two-body operators which are found in the same way. If one begins with

$$\hat{U} = \prod_{i=1}^N \exp(\epsilon_{\alpha_i} \hat{A}_{\alpha_i}) \prod_{j=1}^N \exp(\epsilon_{\beta_j} \hat{A}_{\beta_j}^\dagger), \quad (30)$$

where $\hat{A}_\alpha, \hat{A}_\beta^\dagger$ are defined in (7), then the overlap is

$$\langle \Psi_\alpha | \Psi_\beta \rangle = \left(\prod_{i=1}^N \frac{\partial^2}{\partial \epsilon_{\alpha_i} \partial \epsilon_{\beta_i}} \right) \langle 0 | \hat{U} | 0 \rangle \Big|_{\text{all } \epsilon' \text{'s} = 0} \quad (31)$$

In applying the theorem the matrix algebra is straightforward; keeping only terms linear in ϵ_{α_i} etc. one finds that for the overlap $T(t) = 0$, and

$$\mathbf{U}_{22} = \mathbf{1} + 2 \sum_{ij} \epsilon_{\alpha_i} \epsilon_{\beta_j} \mathbf{A}_{\alpha_i} \mathbf{A}_{\beta_j}^\dagger. \quad (32)$$

Next, insert \mathbf{U}_{22} into (29), use $\det \mathbf{U}_{22} = \exp \text{tr} \ln \mathbf{U}_{22}$ and expand the logarithm to arrive at the generating function

$$\langle 0 | \hat{U} | 0 \rangle = \exp \left(\sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{n} \text{tr} \left(\sum_{ij} \epsilon_{\alpha_i} \epsilon_{\beta_j} \mathbf{A}_{\alpha_i} \mathbf{A}_{\beta_j}^\dagger \right)^n \right). \quad (33)$$

Finally, one applies (31).

The result is constructed from objects of the form $\text{tr} (\mathbf{A}_{\alpha_1} \mathbf{A}_{\beta_1}^\dagger \mathbf{A}_{\alpha_2} \mathbf{A}_{\beta_2}^\dagger \dots \mathbf{A}_{\alpha_k} \mathbf{A}_{\beta_k}^\dagger)$, which we term a *k-contraction* (related to the contractions of *k* phonons in [13]) and represent with

the notation $c_k[\alpha_1, \beta_1; \alpha_2, \beta_2; \dots; \alpha_k, \beta_k]$ (from the order it is clear that the α 's represent final states and the β 's initial states). Then $\langle \Psi_\alpha | \Psi_\beta \rangle$ is a finite sum of products of k -contractions; each term in the sum is a product of m_1 1-contractions, m_2 2-contractions, \dots , and m_N N -contractions, with the $\{m_i\}$ taking all possible values under the constraint that $\sum_k km_k = N$. For example, there will be terms consisting of N 1-contractions, terms consisting of $N - 2$ 1-contractions and one 2-contraction, and so on, up to terms consisting of 1 N -contraction. The full expression is

$$\langle \Psi_\alpha | \Psi_\beta \rangle = \sum_{\mathcal{P}} \sum_{\ell=0}^N w_\ell^0(\alpha_1, \alpha_2, \dots, \alpha_\ell; \beta_1, \dots, \beta_\ell) ((N - \ell)!)^{-1} \prod_{k=\ell+1}^N c_1[\alpha_k, \beta_k] \quad (34)$$

where \mathcal{P} means the sum is over all permutations of $\{\alpha_i\}$ and $\{\beta_j\}$. The coefficients w_ℓ^0 are given by

$$\begin{aligned} w_\ell^0 \equiv & \sum_{m_2, \dots, m_\ell} z_\ell(m_2, \dots, m_\ell) \\ & \times \underbrace{c_2[\alpha_1, \beta_1; \alpha_2, \beta_2] c_2[\alpha_3, \beta_3; \alpha_4, \beta_4] \dots c_2[\alpha_{m_2-1}, \beta_{m_2-1}; \alpha_{m_2}, \beta_{m_2}]}_{m_2} \\ & \times \underbrace{c_3[\alpha'_1, \beta'_1; \alpha'_2, \beta'_2; \alpha'_3, \beta'_3] \dots c_3[\alpha'_{m_3-2}, \beta'_{m_3-2}; \alpha'_{m_3-1}, \beta'_{m_3-1}; \alpha'_{m_3}, \beta'_{m_3}]}_{m_3} \times \dots \end{aligned} \quad (35)$$

The organization of (34) is such as to ease the interpretation in bosons in the next section.

The coefficients $z_\ell(m_2, \dots, m_\ell)$ are found through the expansion of (33):

$$z_\ell(m_2, \dots, m_\ell) = (-2)^{\ell-M} \prod_{k>1} \frac{1}{k^{m_k} m_k!}, \quad (36)$$

where $\ell = \sum_{k>1} km_k$ and $M = \sum_{k>1} m_k$, and $w_0^0 = z_0 = 1$ (hence the normalization chosen above). Note that we have set $m_1 = N - \ell$ and that the w_ℓ^0 have no explicit dependent on N .

To illustrate, consider a single j -shell fermion space, with $\Omega = j + \frac{1}{2}$, and further consider only the fermion pair with total $J = 0$: $A_0^\dagger = (2)^{-1/2} [a_j^\dagger \otimes a_j^\dagger]_0$; the matrix \mathbf{A}_0^\dagger is antidiagonal, $(\mathbf{A}_0^\dagger)_{mm'} = (2\Omega)^{-1/2} (-1)^{j-m} \delta_{m, -m'}$. Then the k -contraction $c_k = \text{tr} (\mathbf{A}_0 \mathbf{A}_0^\dagger)^k = (2\Omega)^{-k+1}$. Following the above definitions,

$$z_2(1) = -1, \quad z_3(0, 1) = 4/3, \quad z_4(2, 0, 0) = 1/2,$$

$$z_4(0, 0, 1) = -2, z_5(1, 1, 0, 0) = -4/3, z_5(0, 0, 0, 1) = 16/5, \text{ etc.}$$

and

$$w_2^0 = -\frac{1}{2\Omega}, w_3^0 = \frac{1}{3\Omega^2}, w_4^0 = \frac{1}{8\Omega^2} - \frac{1}{4\Omega^3}, w_5^0 = -\frac{1}{6\Omega^3} + \frac{1}{5\Omega^4}. \quad (37)$$

Then from equation (34), including the sum over all permutations which gives a factor $N!$, one obtains for example the norm of the state with 5 $J = 0$ pairs:

$$\langle 0 | A_0^5 (A_0^\dagger)^5 | 0 \rangle = 5! \left(1 - \frac{10}{\Omega} + \frac{35}{\Omega^2} - \frac{50}{\Omega^3} + \frac{24}{\Omega^4} \right),$$

which agrees with the general result from the commutation relation $[A_0, A_0^\dagger] = 1 - \hat{N}/\Omega$ (\hat{N} is the fermion number operator), that is, the norm is $N!\Omega!/(\Omega - N)!\Omega^{N-1}$.

In the same way as for the overlap one can derive the matrix elements for one- and two-body operators $\hat{\mathcal{O}}^{1,2}$ (and for general n -body operator, if so desired):

$$\langle \Psi_\alpha | \hat{\mathcal{O}}^{1,2} | \Psi_\beta \rangle = \sum_{\mathcal{P}} \sum_{\ell=1}^N ((N - \ell)!)^{-1} \sum_{k=1}^{\ell} \tilde{w}_k^{1,2}[\mathcal{O}^{1,2}; \alpha_1, \dots, \beta_k] w_{\ell-k}^0 \prod_{m=\ell+1}^N c_k[\alpha_k, \beta_k]. \quad (38)$$

For a one-body operator \hat{T} we define $\mathbf{A}_\alpha \mathbf{T} = \mathbf{A}_{\alpha(T)}$; then

$$\tilde{w}_k^1(T; \alpha_1, \alpha_2, \dots, \beta_k) \equiv 2(-2)^{k-1} c_k[\alpha_1(T), \beta_1; \alpha_2, \beta_2; \dots; \alpha_k, \beta_k]. \quad (39)$$

For a two-body operator \hat{V}

$$\begin{aligned} & \tilde{w}_k^2(V; \alpha_1, \alpha_2, \dots, \beta_k) \equiv \\ & \sum_{\mu\nu} \langle \mu | \hat{V} | \nu \rangle (-2)^{k-1} \sum_{l=1}^k (c_l[\mu, \beta_1; \alpha_2, \beta_2; \dots; \alpha_l, \beta_l] c_{k-l+1}[\alpha_1, \nu; \alpha_{l+1}, \beta_{l+1}; \dots; \alpha_k, \beta_k] \\ & - 2(\delta_{l,1} - 1) c_{k+1}[\mu, \beta_1; \alpha_2, \beta_2; \dots; \alpha_l, \beta_l; \alpha_1, \nu; \alpha_{l+1}, \beta_{l+1}; \dots; \alpha_k, \beta_k]). \end{aligned} \quad (40)$$

Returning to our illustration, first consider the the number operator \hat{N} , which is represented by just the unit matrix; in this case, again restricting ourselves to $J = 0$ pairs, $\tilde{w}_k^1 = 2/(-\Omega)^{k-1}$. Similarly for the pairing interaction $V = A_0^\dagger A_0$,

$$\tilde{w}_k^2 = \left(\frac{1}{-\Omega} \right)^{k-1} \left(k \left(1 - \frac{1}{\Omega} \right) + \frac{1}{\Omega} \right).$$

The reader is invited to check that these coefficients reproduce the correct matrix elements.

Therefore, given any two states constructed from fermion pairs and the matrices representing those constituent pairs, the above formulas give exactly the overlap and the matrix element for one- and two-body operators. For a pair-condensate wavefunction, the matrix elements can be found quickly through recursion [13,14].

Throughout this paper we will discuss the effect of truncating the boson Fock space, that is taking a restricted number of boson species from which to construct states and operators, on boson mappings. But which boson species should we keep? In regards to this question we merely wish to comment that Rowe, Song and Chen give a variational principle [14] which seems useful in this regard, and is probably related in some approximation to the Hartree-Fock-Bogoliubov states that Otsuka and Yoshinaga [16] use in their mapping of deformed nuclei.

IV. BOSON REPRESENTATIONS OF FERMION MATRIX ELEMENTS

We now want to translate the fermion matrix elements into boson space. We take the simple mapping of fermion states into boson states

$$|\Psi_\beta\rangle \rightarrow |\Phi_\beta\rangle = \prod_{m=1}^N b_{\beta_m}^\dagger |0\rangle, \quad (41)$$

where the b^\dagger are boson creation operators. We construct boson operators that preserve matrix elements, introducing boson operators $\hat{\mathcal{T}}_B$, $\hat{\mathcal{V}}_B$, and most importantly the *norm operator* $\hat{\mathcal{N}}_B$ such that $(\Phi_\alpha|\hat{\mathcal{T}}_B|\Phi_\beta) = \langle\Psi_\alpha|\hat{T}|\Psi_\beta\rangle$, $(\Phi_\alpha|\hat{\mathcal{V}}_B|\Phi_\beta) = \langle\Psi_\alpha|\hat{V}|\Psi_\beta\rangle$. and $(\Phi_\alpha|\hat{\mathcal{N}}_B|\Phi_\beta) = \langle\Psi_\alpha|\Psi_\beta\rangle$. We term $\hat{\mathcal{T}}_B, \hat{\mathcal{V}}_B$ the boson *representations* of the fermion operators \hat{T} , \hat{V} . The boson basis $|\Phi_\beta\rangle$ is an orthogonal basis. The fermion norm operator in the boson space will be given by

$$\hat{\mathcal{N}}_B = \sum \frac{\langle\Psi_\alpha|\Psi_\beta\rangle}{(N!)^2} b_{\alpha_1}^\dagger \dots b_{\alpha_N}^\dagger b_{\beta_1} \dots b_{\beta_N} \quad (42)$$

where the coefficients are given by (34). Because the matrices $\mathbf{A}_\alpha, \mathbf{A}_\beta^\dagger$ are orthogonal (see (8)), the one-contraction is simply $c_1[\alpha_k, \beta_k] = \delta_{\alpha_k, \beta_k}$. Using the fact that $b_\alpha^\dagger b_\alpha = \hat{N}$, the

number operator, we find the ‘linked-cluster’ (à la Kishimoto and Tamura [17,18] although with differences) expansion of the representations to be of the form

$$\hat{\mathcal{N}}_B = 1 + \sum_{\ell=2}^{\infty} \sum_{\{\sigma, \tau\}} w_{\ell}^0(\sigma_1, \dots, \sigma_{\ell}; \tau_1, \dots, \tau_{\ell}) \prod_{i=1}^{\ell} b_{\sigma_i}^{\dagger} \prod_{j=1}^{\ell} b_{\tau_j}. \quad (43)$$

and similarly for $\hat{\mathcal{V}}_B, \hat{\mathcal{T}}_B$. In the norm operator the ℓ -body terms express the fact that the fermion-pair operators do not have exactly bosonic commutation relations, and act to enforce the Pauli principle.

In the example of a single j -shell given in the previous section, using the coefficients (37) and with the mapping $A_0^{\dagger} \rightarrow s^{\dagger}$, the purely s -boson part of the norm is

$$\hat{\mathcal{N}}_B = 1 - \frac{1}{2\Omega} s^{\dagger} s^{\dagger} s s + \frac{1}{3\Omega^2} s^{\dagger} s^{\dagger} s^{\dagger} s s s + \left(\frac{1}{8\Omega^2} - \frac{1}{4\Omega^3} \right) s^{\dagger} s^{\dagger} s^{\dagger} s^{\dagger} s s s s + \dots \quad (44)$$

which again yields the correct matrix elements.

The norm operator can be conveniently and compactly expressed in terms of bosons by using the fermion norm matrix (31) in terms of the fermion generating function (33). Taking derivatives with respect to the ϵ_{β_i} in (31) is like contracting $b_{\beta_i}^{\dagger}$ with a b_{β_i} , and with respect to ϵ_{α_i} is like contracting a b_{α_i} with a $b_{\alpha_i}^{\dagger}$. Hence the norm matrix is just the generating function (33) with $\epsilon_{\alpha_i} \rightarrow b_{\alpha_i}^{\dagger}$ and $\epsilon_{\beta_j} \rightarrow b_{\beta_j}$. However, since these bosons do not commute with one another, we must take the normal order:

$$\hat{\mathcal{N}}_B = : \exp \left(-\frac{1}{2} \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \hat{C}_k \right) : \quad (45)$$

where the colons ‘:’ refer to normal-ordering of the boson operators, and $\hat{C}_k = 2: \text{tr } \mathbf{P}^k :$ is the k th-order Casimir of $\text{SU}(2\Omega)$, with $\mathbf{P} = \sum b_{\sigma}^{\dagger} b_{\tau} \mathbf{A}_{\sigma} \mathbf{A}_{\tau}^{\dagger}$ (the trace is over the matrices and not the boson Fock space). This norm operator, which takes into account the exchange terms in the BZ expansion of a fermion pair given in (14), is found in Ref. [19].

Similarly — and this is a new result we have not seen elsewhere in the literature — the representations $\hat{\mathcal{T}}_B, \hat{\mathcal{V}}_B$ can also be written in compact form:

$$\hat{\mathcal{T}}_B = 2 \sum_{\sigma, \tau} : \text{tr} \left[\mathbf{A}_{\sigma} \mathbf{T} \mathbf{A}_{\tau}^{\dagger} \mathbf{G} \right] b_{\sigma}^{\dagger} b_{\tau} \hat{\mathcal{N}}_B : \quad (46)$$

$$\begin{aligned}\hat{\mathcal{V}}_B = & \sum_{\mu,\nu} \langle \mu | V | \nu \rangle \sum_{\sigma,\tau} \left\{ \text{tr} \left[\mathbf{A}_\sigma \mathbf{A}_\mu^\dagger \mathbf{G} \right] \text{tr} \left[\mathbf{A}_\nu \mathbf{A}_\tau^\dagger \mathbf{G} \right] \right. \\ & \left. + 4 \text{tr} \left[\mathbf{A}_\sigma \mathbf{A}_\mu^\dagger \mathbf{P} \mathbf{G} \mathbf{A}_\mu \mathbf{A}_\tau^\dagger \mathbf{G} \right] \right\} b_\sigma^\dagger b_\tau \hat{\mathcal{N}}_B, \end{aligned} \quad (47)$$

where $\mathbf{G} = (\mathbf{1} + 2\mathbf{P})^{-1}$. These compact forms are useful for formal manipulation. Furthermore they have the powerful property of exactly expressing the fermion matrix elements under *any* truncation, a fact not previously appreciated in the literature even for the norm operator [19]. By this we mean the following: suppose we truncate our fermion Fock space to states constructed from a restricted set of pairs $\{\bar{\sigma}\}$. Such a truncation need *not* correspond to any subalgebra. Then the representations in the corresponding truncated boson space, which still exactly reproduce the fermion matrix elements and which we denote by $[\mathcal{N}_B]_T$ etc., are the same as those given above, retaining only the ‘allowed’ bosons with unrenormalized coefficients. For example

$$[\mathcal{N}_B]_T =: \exp \left(-\frac{1}{2} \sum_{k=2}^{\infty} \frac{(-1)^k}{k} [\hat{C}_k]_T \right) : \quad (48)$$

where

$$[\hat{C}_k]_T = 2: \text{tr} ([\mathbf{P}]_T)^k : , \quad [\mathbf{P}]_T = \sum_{\bar{\sigma}\bar{\tau}} b_{\bar{\sigma}}^\dagger b_{\bar{\tau}} \mathbf{A}_{\bar{\sigma}} \mathbf{A}_{\bar{\tau}}^\dagger. \quad (49)$$

This invariance of the coefficients under truncation will not hold true for the boson *images* introduced below.

With the boson representations of fermion operators in hand, one can express the fermion Schrödinger equation (1) with $\hat{H} = \hat{T} + \hat{V}$ as a generalized boson eigenvalue equation,

$$\hat{\mathcal{H}}_B |\Phi_\lambda\rangle = E_\lambda \hat{\mathcal{N}}_B |\Phi_\lambda\rangle. \quad (50)$$

Here $\hat{\mathcal{H}}_B$ is the boson representation of the fermion Hamiltonian. Every physical fermion eigenstate in (1) has a corresponding eigenstate, with the same eigenvalue, in (50). Because the space of states constructed from pairs of fermions is overcomplete, there also exist spurious boson states that do not correspond to unique physical fermion states. These spurious states will have zero eigenvalues and so can be identified.

V. BOSON IMAGES

In general the boson representations given in (45), (46) and (47) do not have good convergence properties, so that simple termination of the series such as (43) in ℓ -body terms is impossible and use of the generalized eigenvalue equation (50), as written, is problematic. Instead we “divide out” the norm operator to obtain the *boson image*, i.e. schematically,

$$\hat{h} \sim “\hat{\mathcal{H}}_B/\hat{\mathcal{N}}_B.” \quad (51)$$

That this is reasonable is suggested by the explicit forms of (46) and (47). The hope of course is that \hat{h} is finite or nearly so, so that a 1+2-body fermion Hamiltonian is mapped to an image

$$\hat{h} \sim \theta_1 b^\dagger b + \theta_2 b^\dagger b^\dagger b b + \theta_3 b^\dagger b^\dagger b^\dagger b b b + \theta_4 b^\dagger b^\dagger b^\dagger b^\dagger b b b b + \dots \quad (52)$$

with the ℓ -body terms, $\ell > 2$, zero or greatly suppressed. We now discuss how to “divide out” the norm.

A. Exact results: Full Space

It turns out that for a number of cases the image of the Hamiltonian is exactly finite. In particular, for the full boson Fock space the representations factor in a simple way: $\hat{T}_B = \hat{\mathcal{N}}_B \hat{T}_B = \hat{T}_B \hat{\mathcal{N}}_B$ and $\hat{V}_B = \hat{\mathcal{N}}_B \hat{V}_B = \hat{V}_B \hat{\mathcal{N}}_B$, where the factored operators \hat{T}_B , \hat{V}_B , which we term the boson images of \hat{T} , \hat{V} , commute with the norm operator and have simple forms:

$$\hat{T}_B = 2 \sum_{\sigma\tau} \text{tr} \left(\mathbf{A}_\sigma \mathbf{T} \mathbf{A}_\tau^\dagger \right) b_\sigma^\dagger b_\tau, \quad (53)$$

$$\hat{V}_B = \sum_{\mu\nu} \langle \mu | V | \nu \rangle \left[b_\mu^\dagger b_\nu + 2 \sum_{\sigma\sigma'} \sum_{\tau\tau'} \text{tr} \left(\mathbf{A}_\sigma \mathbf{A}_\mu^\dagger \mathbf{A}_{\sigma'} \mathbf{A}_\tau^\dagger \mathbf{A}_\nu \mathbf{A}_{\tau'}^\dagger \right) b_\sigma^\dagger b_{\sigma'}^\dagger b_\tau b_{\tau'} \right] \quad (54)$$

The proof of the factorization and commutation requires use of the identities

$$2 \sum_{\alpha} \text{tr}(\mathbf{Q} \mathbf{A}_{\alpha}^{\dagger}) \text{tr}(\mathbf{A}_{\alpha} \mathbf{R}) = \text{tr}(\mathbf{Q} \mathbf{R}) - \text{tr}(\mathbf{Q}^T \mathbf{R}), \quad (55)$$

$$2 \sum_{\alpha} \text{tr}(\mathbf{Q} \mathbf{A}_{\alpha}^{\dagger} \mathbf{R} \mathbf{A}_{\alpha}) = \text{tr}(\mathbf{Q}) \text{tr}(\mathbf{R}) - \text{tr}(\mathbf{Q}^T \mathbf{R}) \quad (56)$$

(\mathbf{Q}^T is the transpose of \mathbf{Q}) which in turn are proved using the completeness relation (9). The image Hamiltonian $\hat{H}_B = \hat{T}_B + \hat{V}_B$ is the one determined by BZ if one decomposes the Hamiltonian into multipole-multipole form and then maps these multipole operators. As discussed earlier, these BZ multipole operators are finite in the full space. This result, and its relation to other mappings, was noted by Marshalek [8,10]

Thus any boson representation of a Hamiltonian factorizes: $\hat{\mathcal{H}}_B = \hat{\mathcal{N}}_B \hat{H}_B$ in the full space. Since the norm operator is a function of the $\text{SU}(2\Omega)$ Casimir operators it commutes with the boson images of fermion operators, and one can simultaneously diagonalize both $\hat{\mathcal{H}}_B$ and $\hat{\mathcal{N}}_B$. Then Eqn. (50) becomes

$$\hat{H}_B |\Phi_{\lambda}\rangle = E'_{\lambda} |\Phi_{\lambda}\rangle. \quad (57)$$

where $E'_{\lambda} = E_{\lambda}$ for the physical states, but E'_{λ} for the spurious states is no longer necessarily zero. The boson Hamiltonian \hat{H}_B is by construction Hermitian and, if one starts with at most only two-body interactions between fermions, has at most two-body boson interactions. All physical eigenstates of the original fermion Hamiltonian will have counterparts in (57). It should be clear that transition amplitudes between physical eigenstates will be preserved. Spurious states will also exist but, since the norm operator $\hat{\mathcal{N}}_B$ commutes with the boson image Hamiltonian \hat{H}_B , the physical eigenstates and the spurious states will not admix. Also the spurious states can be identified because, while they will no longer have zero energy eigenvalues, they have eigenvalue zero with respect to the norm operator.

B. Exact Results: Truncated space

The boson Schrödinger equation (57), though finite, is not of much use as the boson Fock space is much larger than the original fermion Fock space, and we still must truncate the boson Fock space. Although the representations remain exact under truncation, the

factorization into the image does not persist in general: $[\hat{\mathcal{H}}_B]_T \neq [\hat{\mathcal{N}}_B]_T [\hat{H}_B]_T$. Again for example consider the pairing interaction, even in multiple j -shells, and a truncation to just s -bosons; then $[\hat{H}_B]_T = s^\dagger s + \frac{1}{2\Omega} s^\dagger s^\dagger s s$ whereas the appropriate image is actually $\hat{h} = s^\dagger s - \frac{1}{\Omega} s^\dagger s^\dagger s s$. This was recognized by Marshalek [10]. (An alternate formulation [10] does not require the complete Fock space, but mixes physical and spurious states and so always requires a projection operator.)

If the truncated set $\{\bar{\alpha}\}$ represents a closed subalgebra, that is, if the truncated set of fermion pairs are closed under double commutations:

$$[A_{\bar{\alpha}}, A_{\bar{\beta}}^\dagger] = \delta_{\bar{\alpha}, \bar{\beta}} - T_{\bar{\alpha}\bar{\beta}}, \quad (58)$$

$$[T_{\bar{\alpha}\bar{\beta}}, A_{\bar{\gamma}}^\dagger] = \sum_{\bar{\sigma}} \Gamma_{\bar{\alpha}\bar{\beta}; \bar{\gamma}}^{\bar{\sigma}} A_{\bar{\sigma}}^\dagger; \quad (59)$$

then a factorization [19]

$$[\hat{\mathcal{H}}_B]_T = [\hat{\mathcal{N}}_B]_T \hat{h}_D = \hat{h}_D^\dagger [\hat{\mathcal{N}}_B]_T \quad (60)$$

does exist, with \hat{h}_D at most two-body, but not necessarily Hermitian:

$$(T_{\bar{\alpha}\bar{\beta}})_D = \sum_{\bar{\sigma}\bar{\tau}} \Gamma_{\bar{\alpha}\bar{\beta}; \bar{\tau}}^{\bar{\sigma}} b_{\bar{\sigma}}^\dagger b_{\bar{\tau}}, \quad (61)$$

$$(A_{\bar{\mu}}^\dagger A_{\bar{\nu}})_D = b_{\bar{\mu}}^\dagger \left(b_{\bar{\nu}} - \sum_{\bar{\sigma}\bar{\tau}\bar{\tau}'} \Gamma_{\bar{\nu}\bar{\tau}'; \bar{\tau}}^{\bar{\sigma}} b_{\bar{\sigma}}^\dagger b_{\bar{\tau}} b_{\bar{\tau}'} \right). \quad (62)$$

We term this a generalized *Dyson* image [8,9,11]. Although $\langle \text{physical} | \hat{h}_D | \text{spurious} \rangle = 0$, if \hat{h}_D is non-Hermitian, that is $[[\hat{\mathcal{N}}_B]_T, \hat{h}_D] \neq 0$, then unfortunately $\langle \text{spurious} | \hat{h}_D | \text{physical} \rangle \neq 0$. In the full space, of course, all definitions of boson images coincide and yield the same result.

We have found conditions under which \hat{h}_D is additionally Hermitian and commutes with the truncated norm operator $[\hat{\mathcal{N}}_B]_T$. Consider a partition of the single fermion states labeled by $i = (i_a, i_c)$, where the dimension of each subspace is $2\Omega_a, 2\Omega_c$ so that $\Omega = 2\Omega_a\Omega_c$. We denote the amplitudes for the truncated space as $\mathbf{A}_\alpha^\dagger$ and assume they can be factored, $(\mathbf{A}_\alpha^\dagger)_{ij} = (\mathbf{K}^\dagger)_{i_a j_a} \otimes (\bar{\mathbf{A}}_\alpha^\dagger)_{i_c j_c}$, with $\mathbf{K}^\dagger \mathbf{K} = \mathbf{K} \mathbf{K}^\dagger = \frac{1}{2\Omega_a}$ and $\mathbf{K}^T = (-1)^p \mathbf{K}$, where $p = 0$

(symmetric) or $p = 1$ (antisymmetric). Furthermore we assume the completeness relation (9), which was crucial for proving that $\hat{\mathcal{H}}_B = \hat{\mathcal{N}}_B \hat{H}_B$ [20], is valid for the truncated space; i.e.,

$$\sum_{\bar{\alpha}} (\bar{\mathbf{A}}_{\bar{\alpha}}^{\dagger})_{i_c j_c} (\bar{\mathbf{A}}_{\bar{\alpha}})_{j'_c i'_c} = \frac{1}{2} \left[\delta_{i_c, i'_c} \delta_{j_c, j'_c} - (-1)^p \delta_{i_c, j'_c} \delta_{i'_c, j_c} \right]. \quad (63)$$

The norm operator in the truncated space then becomes

$$[\hat{\mathcal{N}}_B]_T =: \exp \sum_{k=2} \left(\frac{-1}{\Omega_a} \right)^{k-1} \frac{1}{k} \text{tr}(\bar{\mathbf{P}}^k), \quad (64)$$

where $\bar{\mathbf{P}} = \sum_{\bar{\sigma}\bar{\tau}} b_{\bar{\sigma}}^{\dagger} b_{\bar{\tau}} \bar{\mathbf{A}}_{\bar{\sigma}} \bar{\mathbf{A}}_{\bar{\tau}}^{\dagger}$ so that $[\mathbf{P}]_T = \left(\frac{1}{2\Omega_a} \right) \bar{\mathbf{P}}$. In this case the boson image of a one-body operator is the truncation of the boson image in the full space,

$$[\hat{\mathcal{T}}_B]_T = [\hat{\mathcal{N}}_B]_T [\hat{T}_B]_T \quad (65)$$

$$[\hat{T}_B]_T = 2 \sum_{\bar{\sigma}, \bar{\tau}} \text{tr} \left(\mathbf{A}_{\bar{\sigma}} \mathbf{T} \mathbf{A}_{\bar{\tau}}^{\dagger} \right) b_{\bar{\sigma}}^{\dagger} b_{\bar{\tau}}. \quad (66)$$

The representation of a two-body interaction can be factored into a boson image times the truncated norm,

$$[\hat{\mathcal{V}}_B]_T = [\hat{\mathcal{N}}_B]_T \hat{v}_D; \quad (67)$$

however, \hat{v}_D , while finite (1+2-body), is not simply related to $[\hat{V}_B]_T$ as is the case for one-body operators. If one writes

$$\hat{v}_D = \sum_{\bar{\sigma}, \bar{\tau}} \langle \bar{\sigma} | V | \bar{\tau} \rangle b_{\bar{\sigma}}^{\dagger} b_{\bar{\tau}} + \sum_{\bar{\sigma}\bar{\sigma}'\bar{\tau}\bar{\tau}'} \langle \bar{\sigma}\bar{\sigma}' | v | \bar{\tau}\bar{\tau}' \rangle b_{\bar{\sigma}}^{\dagger} b_{\bar{\sigma}'}^{\dagger} b_{\bar{\tau}} b_{\bar{\tau}'}, \quad (68)$$

then matrix elements of the two-boson interaction are

$$\begin{aligned} & \langle \bar{\sigma}\bar{\sigma}' | v | \bar{\tau}\bar{\tau}' \rangle = \\ & \sum_{\mu, \nu} \frac{\langle \mu | V | \nu \rangle}{\Omega_a (2\Omega_a - (-1)^p) (\Omega_a + (-1)^p)} \text{tr}_a \{ \text{tr}_c (\bar{\mathbf{A}}_{\bar{\sigma}} \bar{\mathbf{A}}_{\bar{\tau}}^{\dagger} \mathbf{A}_{\nu} \bar{\mathbf{A}}_{\bar{\tau}'}^{\dagger}) \text{tr}_c (\bar{\mathbf{A}}_{\bar{\sigma}'} \mathbf{A}_{\mu}^{\dagger}) \\ & \quad + 2\Omega_a [\text{tr}_c (\bar{\mathbf{A}}_{\bar{\sigma}} \mathbf{A}_{\mu}^{\dagger} \bar{\mathbf{A}}_{\bar{\sigma}'} \bar{\mathbf{A}}_{\bar{\tau}}^{\dagger} \mathbf{A}_{\nu} \bar{\mathbf{A}}_{\bar{\tau}'}^{\dagger}) - \text{tr}_c (\bar{\mathbf{A}}_{\bar{\sigma}} \bar{\mathbf{A}}_{\mu}^{\dagger} \bar{\mathbf{A}}_{\bar{\sigma}'} \bar{\mathbf{A}}_{\bar{\tau}}^{\dagger} \bar{\mathbf{A}}_{\bar{\nu}} \mathbf{K} \bar{\mathbf{A}}_{\bar{\tau}'}^{\dagger})] \\ & \quad - \Omega_a (2\Omega_a + (-1)^p) \text{tr}_c (\mathbf{A}_{\nu} \mathbf{K}^{\dagger} \bar{\mathbf{A}}_{\bar{\tau}}^{\dagger} \bar{\mathbf{A}}_{\bar{\sigma}'} \bar{\mathbf{A}}_{\bar{\tau}'}^{\dagger}) \delta_{\bar{\sigma}, \mu} \}, \end{aligned} \quad (69)$$

where tr_a, tr_c are traces only in the a - and c - spaces, respectively.

Upon inspection one sees the image (69) is not constrained to be Hermitian. Consider the additional condition between the matrix elements of the interaction:

$$\begin{aligned} & \sum_{\mu, \nu} \langle \mu | V | \nu \rangle \sum_{i_a, j_a} (\mathbf{A}_\nu)_{i_a i_c, j_a j_c} (\mathbf{A}_\mu^\dagger)_{j_a j'_c, i_a i'_c} \\ &= N_a \sum_{\mu, \nu} \langle \mu | V | \nu \rangle \sum_{i_a, j_a} (\mathbf{A}_\nu)_{i_a i_c, j_a j_c} (\mathbf{K}^\dagger)_{j_a, i_a} \sum_{i'_a, j'_a} (\mathbf{K})_{i'_a, j'_a} (\mathbf{A}_\mu^\dagger)_{j'_a j'_c, i'_a i'_c} \end{aligned} \quad (70)$$

where the factor $N_a = \Omega_a(2\Omega_a + (-1)^p)$ is the number of pairs in the excluded subspace. While condition (70) looks complicated there are interactions that satisfy it; for example, two-body interactions constructed from one-body operators $\hat{V} = \hat{T}_{\bar{\alpha}\bar{\beta}} \hat{T}_{\bar{\alpha}'\bar{\beta}'}$ where $\hat{T}_{\bar{\alpha}\bar{\beta}} = [A_{\bar{\alpha}}^\dagger, A_{\bar{\beta}}]$. When (70) is satisfied then \hat{v}_D is Hermitian and although $\hat{v}_D \neq [\hat{V}_B]_T$ they are simply related:

$$\begin{aligned} \hat{v}_D &= \sum_{\bar{\sigma}, \bar{\tau}} \langle \bar{\sigma} | V | \bar{\tau} \rangle b_{\bar{\sigma}}^\dagger b_{\bar{\tau}} \\ &+ 2f_{\Omega_a} \sum_{\mu, \nu} \langle \mu | V | \nu \rangle \sum_{\bar{\sigma}\bar{\sigma}', \bar{\tau}\bar{\tau}'} \text{tr} \left(\mathbf{A}_{\bar{\sigma}} \mathbf{A}_\mu^\dagger \mathbf{A}_{\bar{\sigma}'} \mathbf{A}_{\bar{\tau}}^\dagger \mathbf{A}_\nu \mathbf{A}_{\bar{\tau}'}^\dagger \right) b_{\bar{\sigma}}^\dagger b_{\bar{\sigma}'}^\dagger b_{\bar{\tau}} b_{\bar{\tau}'} \end{aligned} \quad (71)$$

with $f_{\Omega_a} = 4\Omega_a^2/N_a$ renormalizing the two-boson part of $[\hat{V}_B]_T$ by a factor which ranges from unity (full space) to 2 for a very small subspace. Not all interactions satisfy (70); for example, the pairing interaction never does except in the full space. For the pairing interaction $\langle \mu | V^{\text{pairing}} | \nu \rangle = \delta_{\mu,0} \delta_{\nu,0} G$, and $\mathbf{A}_0 \mathbf{A}_0^\dagger = \frac{1}{2\Omega}$, and the image (67) $\hat{v}_D^{\text{pairing}}$ becomes (remembering $\Omega = 2\Omega_a \Omega_c$)

$$G \left\{ \hat{N}_0 \left[1 - \frac{2}{\Omega} \hat{N} + \frac{1}{\Omega} + \frac{\hat{N}_0}{\Omega} \right] + \sum_{\bar{\tau}\bar{\tau}' \neq 0, \bar{\sigma}} \text{tr} \left(\bar{\mathbf{A}}_{\bar{\sigma}} \bar{\mathbf{A}}_{\bar{\tau}}^\dagger \bar{\mathbf{A}}_0 \bar{\mathbf{A}}_{\bar{\tau}'}^\dagger \right) b_{\bar{\sigma}}^\dagger b_0^\dagger b_{\bar{\tau}} b_{\bar{\tau}'} \right\}, \quad (72)$$

where \hat{N} is the total number of bosons, $\hat{N} = \sum_{\bar{\tau}} b_{\bar{\tau}}^\dagger b_{\bar{\tau}}$, and $\hat{N}_0 = b_0^\dagger b_0$. The second term in (72) is not Hermitian but can be transformed away by a similarity transformation [21], leaving the first term as a finite Hermitian image which gives the correct eigenvalues for all N .

The $\text{SO}(8)$ and $\text{Sp}(6)$ models [22] belong to a class of models which have a subspace for which (63) is valid and interactions which satisfy (70). In these models the shell model orbitals have a definite angular momentum $\vec{\mathbf{j}}$ and are partitioned into a pseudo orbital angular

momentum $\vec{\mathbf{k}}$ and pseudospin $\vec{\mathbf{i}}, \vec{\mathbf{j}} = \vec{\mathbf{k}} + \vec{\mathbf{i}}$. The amplitudes are then given as products of Clebsch-Gordon coefficients,

$$\left(A_{\alpha}^{\dagger}\right)_{ij} = \frac{(1 + (-1)^{K+I})}{2} (k m_i, k m_j | K_{\alpha} M_{\alpha}) (i \mu_i, i \mu_j | I_{\alpha} \mu_{\alpha}), \quad (73)$$

where K and I are the total pseudo orbital angular momentum and pseudospin respectively of the pair of nucleons. For the SO(8) model $\mathbf{i} = \frac{3}{2}$ and one considers the subspace of pairs with $K = 0$ ($p = 0$), $(\bar{A}_{\alpha}^{\dagger})_{ij} = \frac{(1+(-1)^I)}{2} (i \mu_i, i \mu_j | I_{\alpha} \mu_{\alpha})$; in the Sp(6) model $\mathbf{k} = 1$ and one considers the subspace with $I = 0$ ($p = 1$), $(\bar{A}_{\alpha}^{\dagger})_{ij} = \frac{(1+(-1)^K)}{2} (k m_i, k m_j | K_{\alpha} M_{\alpha})$. The complicated conditions (70) hold true for important cases, such as the quadrupole-quadrupole and other multipole-multipole interactions in the SO(8) and Sp(6) models (that is, interactions of the generic form $P^r \cdot P^r$ in the notation of [22]) where therefore have Hermitian Dyson images. Not all interactions in these models have Hermitian Dyson images. For example, pairing in any model (see (72)) and, in the SO(8) model, the combination $V^7 = S^{\dagger} S + \frac{1}{4} P^2 \cdot P^2$, where $S = \sqrt{\Omega} A_{J=0}$, $\Omega = 4k + 2$, which is the SO(7) limit. It so happens that these particular cases nonetheless can be brought into finite, Hermitian form as discussed in the next section.

C. Approximate or numerical images

The most general image Hamiltonian one can define is

$$\hat{h} \equiv \mathcal{U} \left[\tilde{\mathcal{N}}_B \right]_T^{-1/2} \left[\hat{\mathcal{H}}_B \right]_T \left[\tilde{\mathcal{N}}_B \right]_T^{-1/2} \mathcal{U}^{\dagger}, \quad (74)$$

which is manifestly Hermitian for any truncation scheme and any interaction, with \mathcal{U} a unitary operator. (Because the norm is a singular operator it cannot be inverted. Instead $\left[\tilde{\mathcal{N}}_B \right]_T^{-1/2}$ is calculated from the norm only in the physical subspace, with the zero eigenvalues which annihilate the spurious states retained. Then \hat{h} does not mix physical and spurious states.) If $\mathcal{U} = 1$, this is the democratic mapping [12]. Again, for the full space $\left[\hat{\mathcal{N}}_B, \hat{H}_B \right] = 0$ and hence $\hat{h} = \hat{h}_D = \hat{H}_B$.

This prescription is, we argue, useful for a practical derivation of boson image Hamiltonians. Ignoring for the moment the unitary transformation \mathcal{U} , consider the expansion (52) of \hat{h} . The operators $[\hat{\mathcal{H}}_B]_T$ and $[\tilde{\mathcal{N}}_B]_T^{-1/2}$ have similar expansions, and by multiplying out (74) one sees immediately that the coefficient θ_ℓ depends only on up to ℓ -body terms in $[\mathcal{H}_B]_T$ and $[\tilde{\mathcal{N}}_B]_T^{-1/2}$, derived from 2ℓ -fermion matrix elements which are tractable for ℓ small. Ideally \hat{h} would have at most two-body terms, and our success in finding finite images in the previous section gives us hope that the high-order many-body terms may be small; at any rate the convergence can be calculated and checked term-by-term. Specifically, consider the convergence of the series (52) as a function of ℓ . A rough estimate is that, for an N -boson Fock space, one can truncate to the ℓ -body terms if for $\ell' > \ell$, $\theta_{\ell'}$ is sufficiently small compared to $\theta_\ell \times (N - \ell)! / (N - \ell)!$; the strictest condition is to require $\theta_{\ell'} \ll \theta_\ell / (\ell' - \ell)!$.

Although we have given in section IV analytic expressions for the coefficients of the boson representations, in practice one only needs the fermion matrix elements, given in section III, for the norm and the Hamiltonian or other operators. The coefficients of the image \hat{h} are then found by numerical induction. Suppose one has the coefficients of \hat{h} up to ℓ -boson terms. One takes the matrix elements of $[\tilde{\mathcal{N}}_B]_T^{-1/2}$ —note that in calculating $[\tilde{\mathcal{N}}_B]_T^{-1/2}$, one *first* truncates and *then* calculates the inverse-square-root; the two operations do not commute!—and of the Hamiltonian or transition operator in the $2(\ell + 1)$ fermion space, and multiply out those matrices as in (74), yielding the matrix elements of \hat{h} in the $\ell + 1$ -boson space. The coefficients $\theta_{\ell+1}$ of \hat{h} in (52) are then uniquely determined, up to the freedom embodied in the unitary transform \mathcal{U} .

The Hermitian image \hat{h} , defined in (74), is related to the Dyson image \hat{h}_D , defined in (60), by a similarity transformation $\mathcal{S} = \mathcal{U} [\tilde{\mathcal{N}}_B]_T^{1/2}$,

$$\hat{h} = \mathcal{S} \hat{h}_D \mathcal{S}^{-1}. \quad (75)$$

The similarity transformation \mathcal{S} orthogonalizes the fermion states $|\Psi_\alpha\rangle$ inasmuch $(\mathcal{S}^{-1})^\dagger \tilde{\mathcal{N}}_B \mathcal{S}^{-1} = 1$ in the physical space (and $= 0$ in the spurious space). This is akin to Gram-Schmidt orthogonalization and the freedom to choose \mathcal{U} , and \mathcal{S} , corresponds to

the freedom one has in ordering the vectors in the Gram-Schmidt procedure. The OAI and democratic mappings are just two particular choices out of many; the former orders the states by seniority whereas the latter takes $\mathcal{U} = 1$.

We can use this freedom in the choice of \mathcal{U} to our advantage, which we illustrate in the SO(8) model [22], truncating to the space of $K = 0$ pairs (see eqn. (73)), for which one has only an $I = J = 0$ pair (mapped to an s boson) and a quintuplet of $I = J = 2$ pairs (mapped to d_m bosons, $m = -2 \dots 2$). To second order, the norm is

$$\begin{aligned} \hat{\mathcal{N}}_B = 1 - \frac{1}{2\Omega} s^\dagger s^\dagger s s - \frac{1}{2\Omega} \sum_{L=0,2,4} (2 - 5\delta_{L,0}) [d^\dagger \otimes d^\dagger]_L \cdot [\tilde{d} \otimes \tilde{d}]_L \\ - \frac{\sqrt{5}}{2\Omega} \{ s^\dagger s^\dagger [\tilde{d} \otimes \tilde{d}]_0 + [d^\dagger \otimes d^\dagger]_0 s s \} - \frac{2}{\Omega} \hat{n}_d \hat{n}_s. \end{aligned} \quad (76)$$

We pay particular attention to three interactions which correspond to algebraic limits: the pure pairing interaction $V^{\text{pairing}} = S^\dagger S = \Omega A_0^\dagger A_0$, the quadrupole-quadrupole interaction V^{QQ} ($= P^2 \cdot P^2$ in the notation of [22]), which can be written in terms of SO(6) Casimir operators, and the linear combination of pairing and quadrupole $V^7 = V^{\text{pairing}} + \frac{1}{4} V^{QQ}$ which can be written in terms of SO(7) Casimirs. As discussed in the previous section, the Dyson image of V^{QQ} is Hermitian and finite, and hence $\hat{h}_D = \hat{h}$ with $\mathcal{U} = 1$:

$$(V^{QQ})_D = 20s^\dagger s + 4d^\dagger \cdot \tilde{d} + 4\sqrt{5} \{ s^\dagger s^\dagger [\tilde{d} \otimes \tilde{d}]_0 + [d^\dagger \otimes d^\dagger]_0 s s \} + 8\hat{n}_d \hat{n}_s. \quad (77)$$

The Dyson images of the pairing and SO(7) interactions are finite but non-Hermitian:

$$(V^{\text{pairing}})_D = \Omega s^\dagger s - s^\dagger s^\dagger s s - \sqrt{5} s^\dagger s^\dagger [\tilde{d} \otimes \tilde{d}]_0 - 2\hat{n}_d \hat{n}_s, \quad (78)$$

$$(V^7)_D = (\Omega + 5) s^\dagger s + d^\dagger \cdot \tilde{d} - s^\dagger s^\dagger s s + \sqrt{5} [d^\dagger \otimes d^\dagger]_0 s s. \quad (79)$$

We arrived at these images by computing the fermion matrix elements using the methods of [22] and writing the norm and the representations of the interactions as matrices, which in second order are at most 2×2 , and then directly multiplied the matrices $\mathcal{N}^{-1} \mathcal{H}$. We went to third order to confirm the Dyson images are finite. Using these same matrices we could also calculate $\mathcal{N}^{-1/2} \mathcal{H} \mathcal{N}^{-1/2}$, taking $\mathcal{U} = 1$, which yields again (77) for V^{QQ} ; for V^{pairing} and V^7

the images are then Hermitian but with nonzero third-order, and presumably higher-order, terms.

We then found \mathcal{U} 's $\neq 1$ for both the pairing and SO(7) cases (but not the same \mathcal{U}) such that their respective Hermitian images \hat{h} are finite; the one for pairing is unsurprisingly the OAI prescription, while that for SO(7) is the opposite, orthogonalizing states of low seniority against states of higher seniority. These finite, Hermitian images are:

$$v^{\text{pairing}} = \Omega s^\dagger s - s^\dagger s^\dagger s s, \quad (80)$$

$$v^7 = (\Omega + 5) s^\dagger s + d^\dagger \cdot \tilde{d} - s^\dagger s^\dagger s s. \quad (81)$$

Note that these general Hermitian images are not simply related to the truncation of the full image through an overall renormalization, as is the Hermitian Dyson image (71). For the pairing interaction, the coefficient of the $s^\dagger s^\dagger s s$ term changes sign, while off-diagonal terms such as $d^\dagger d^\dagger s s$, which exist in the image in the full space, vanish, and in some cases like SO(8) other diagonal terms such as $d^\dagger d^\dagger \tilde{d} \tilde{d}$ can also vanish (depending on the model space).

If one uses an “inappropriate” transform \mathcal{S} it can induce an unneeded and unwanted many-body dependence. This principle we illustrate in the SO(8) model, by deriving up to second order the Hermitian image of V^7 but using \mathcal{U}_{OAI} :

$$\begin{aligned} v_{OAI}^7 = & (\Omega + 5) s^\dagger s + d^\dagger \cdot \tilde{d} - \left(1 + \frac{10/\Omega}{1 - \frac{1}{\Omega}}\right) s^\dagger s^\dagger s s - \frac{3/\Omega}{1 - \frac{1}{\Omega}} [d^\dagger \otimes d^\dagger]_0 \cdot [\tilde{d} \otimes \tilde{d}]_0 \\ & - \frac{2\sqrt{5}}{1 - \frac{1}{\Omega}} \sqrt{\left(1 - \frac{2}{\Omega}\right) \left(1 + \frac{4}{\Omega}\right)} \left\{ s^\dagger s^\dagger [\tilde{d} \otimes \tilde{d}]_0 + [d^\dagger \otimes d^\dagger]_0 s s \right\} \end{aligned} \quad (82)$$

plus higher order terms which we drop; this is equivalent to the standard OAI procedure [5] computed in the 2-boson space. In figure 1 we display the spectra of (81) on the left, which is the exact SO(7) result, and (82) on the right, taking $\Omega = 10$ and $N = 7$. The distortions in the right-hand spectrum from the exact result, such as the overall energy shift and the large perturbation in the third band, indicates the importance of the missing many-body terms. In OAI [5] these many-body terms would appear implicitly in the N -dependence of the coefficients for the two-body terms. From the existence of (81) we see that this strong OAI N -dependence is, for this case at least, a needless complication. Therefore it is possible

that some of the N -dependence of OAI is induced by their choice of orthogonalization and could be minimized with a different choice. We are currently exploring how to exploit this freedom to best effect.

VI. EFFECTIVE REPRESENTATIONS AND IMAGES

So far we have considered the mapping of a Hamiltonian in a truncated fermion space to a boson space. That truncated fermion space, however, may be inadequate for reproducing the spectrum of the full space. Consider a single j -shell fermion space. If the interaction is dominated by pairing, then truncation to just $J = 0, 2$ (s, d) pairs is reasonable [5]; but for other interactions, particularly the quadrupole-quadrupole interaction, the s, d -space is inadequate [23]. To rectify these shortcomings one must introduce an effective interaction theory for boson mappings.

An important issue is at what stage to introduce effective operators. For example, one could start from an effective fermion Hamiltonian. Because the fermion-pair basis is non-orthogonal, however, computation of the effective fermion interaction would be tricky; furthermore, the starting Hamiltonian would have a number dependence which would be difficult to separate from the number dependence induced by the subsequent boson mapping. At the other extreme, Sakamoto and Kishimoto [18] start from a boson image and use perturbation theory to account for excluded states. They do not rigorously derive their effective interaction and it is not clear that they properly account for the exchange terms, etc., included in the norm operator.

A better, intermediate approach, is to calculate the boson image in a larger space—say sdg —and then renormalize the pure sd interaction so as to account for the effect of the g -boson [24]. Ideally, one should start with a sufficient number of species of bosons so as to exactly span the fermion Fock space, and then further truncate the space and renormalize. The number of species required will depend on the number of bosons (or pairs) N , however, and we know of no prescription for determining this set of bosons (except for $N = 1$ when

all are necessary).

We present a rigorous and general approach to effective interactions. Following the usual Feshbach [9] derivation, we partition the *boson* Fock space using P to project out the allowed space and Q its compliment, with $P + Q = 1$, $P^2 = P$, $Q^2 = Q$ and $PQ = QP = 0$. Then the truncated representations are simply

$$[\mathcal{H}_B]_T = P\mathcal{H}_BP, \quad (83)$$

$$[\mathcal{N}_B]_T = P\mathcal{N}_BP, \quad (84)$$

and $|\Phi\rangle_T = P|\Phi\rangle$. Then the generalized eigenvalue equation in the full space becomes

$$\left[\hat{\mathcal{H}}_B\right]_T^{\text{eff}}(E_\lambda)|\Phi_\lambda\rangle_T = E_\lambda \left[\hat{\mathcal{N}}_B\right]_T^{\text{eff}}(E_\lambda)|\Phi_\lambda\rangle_T, \quad (85)$$

with

$$\begin{aligned} \left[\hat{\mathcal{H}}_B\right]_T^{\text{eff}}(E) = & P\hat{\mathcal{H}}_BP + P\hat{\mathcal{H}}_BQ \frac{1}{Q(E\hat{\mathcal{N}}_B - \hat{\mathcal{H}}_B)Q} Q\hat{\mathcal{H}}_BP \\ & - P\hat{\mathcal{H}}_BQ \frac{E}{Q(E\hat{\mathcal{N}}_B - \hat{\mathcal{H}}_B)Q} Q\hat{\mathcal{N}}_BP - P\hat{\mathcal{N}}_BQ \frac{E}{Q(E\hat{\mathcal{N}}_B - \hat{\mathcal{H}}_B)Q} Q\hat{\mathcal{H}}_BP \\ & + P\hat{\mathcal{N}}_BQ \frac{E^2}{Q(E\hat{\mathcal{N}}_B - \hat{\mathcal{H}}_B)Q} Q\hat{\mathcal{N}}_BP + E\mathcal{A}(E), \end{aligned} \quad (86)$$

$$\left[\hat{\mathcal{N}}_B\right]_T^{\text{eff}}(E) = P\hat{\mathcal{N}}_BP + \mathcal{A}(E).$$

One can also in principle construct energy-independent, but non-Hermitian, effective representations [25]. There is some ambiguity in the definition of the effective representations as denoted by $\mathcal{A}(E)$. For example, one could define $\left[\hat{\mathcal{H}}_B\right]_T^{\text{eff}}$ to be simply $P\hat{\mathcal{H}}_BP$ with the remaining terms in (86) absorbed into the definition of $\left[\hat{\mathcal{N}}_B\right]_T^{\text{eff}}(E)$, and in principle $\mathcal{A}(E)$ could be anything at all.

Now consider boson images, where one divides out the norm operator. We suggest that effective operator theory may be more efficient when applied to representations rather than images, by which we mean that the corrections are smaller. Suppose one started with the image Hamiltonian in the full space, H_B as defined previously, and from that constructed an effective image in the usual way,

$$H_B^{\text{eff}} = PH_BP + PH_BQ \frac{1}{Q(E - H_B)Q} QH_BP = [H_B]_T + \Delta H_B(E). \quad (87)$$

Now compare that with the effective image constructed from effective representations,

$$h_B^{\text{eff}}(E) \equiv \left([\tilde{\mathcal{N}}_B]_T^{\text{eff}}(E) \right)^{-1/2} [\mathcal{H}_B]_T^{\text{eff}} \left([\tilde{\mathcal{N}}_B]_T^{\text{eff}}(E) \right)^{-1/2} = \hat{h}_B + \Delta h_B(E) \quad (88)$$

(leaving aside the issue of the choice of an overall unitary transformation \mathcal{U}). Now $\hat{h}_B \neq [H_B]_T$. Which approach is better? In those cases, such as $\text{SO}(8)$ and $\text{Sp}(6)$, where, the P -space decouples completely from the Q space, $\Delta h_B = 0$ but ΔH_B cannot be zero. Hence the corrections Δh_B from using effective representations can be smaller than the corrections ΔH_B determined from performing effective operator theory directly on the image.

We now would like to speculate on the possible use of the ambiguity operator $\mathcal{A}(E)$. In effective operator theory the eigenstates are no longer orthogonal because of the truncation of the model space; this is expressed by the fact that one uses either energy-dependent or non-Hermitian effective interactions. We propose that this non-orthogonality could also be embedded in the choice of $\mathcal{A}(E)$, so that the similarity transform on the basis is now $\mathcal{U}\sqrt{[\hat{\mathcal{N}}_B]_T} \mathcal{A}(E)|\Phi_{\bar{\alpha}}\rangle$. The ambiguity operator $\mathcal{A}(E)$ could be chosen so as to minimize the energy dependence of the final boson image. Although the similarity transformation is now energy-dependent, this would not show up in the calculation of the spectrum, but only in the calculation of effective transition operators. These speculations need to be explored in greater detail.

VII. SUMMARY

In order to investigate rigorous foundations for the phenomenological Interacting Boson Model, we have presented a rigorous microscopic mapping of fermion pairs to bosons, paying special attention to exact mapping of matrix elements, Hermiticity, truncation of the model space, and many-body terms. First we presented new, general and compact forms for boson *representations* that preserve fermion matrix elements. We then considered the boson *image* Hamiltonian which results from “dividing out” the norm from the representation; in the full

boson Fock space the image is always finite and Hermitian; in addition we discussed several analytic cases for truncated spaces where the image is also finite and Hermitian. Next, we gave a prescription which is a generalization of both the OAI and democratic mappings; in the most general case for truncated spaces the Hermitian image Hamiltonian may not be finite but we have demonstrated there is some freedom in the mapping that one could possibly exploit to minimize the many-body terms. This freedom, which manifests itself in a similarity transformation that orders the orthogonalization of the underlying fermion basis, depends on the Hamiltonian. Finally, we discussed effective operator theory for boson mappings.

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FIGURES

FIG. 1. Spectrum of $SO(7)$ interaction, for 7 bosons, in $SO(8)$ model with exact (left) and approximate (right) two-body boson Hamiltonians.

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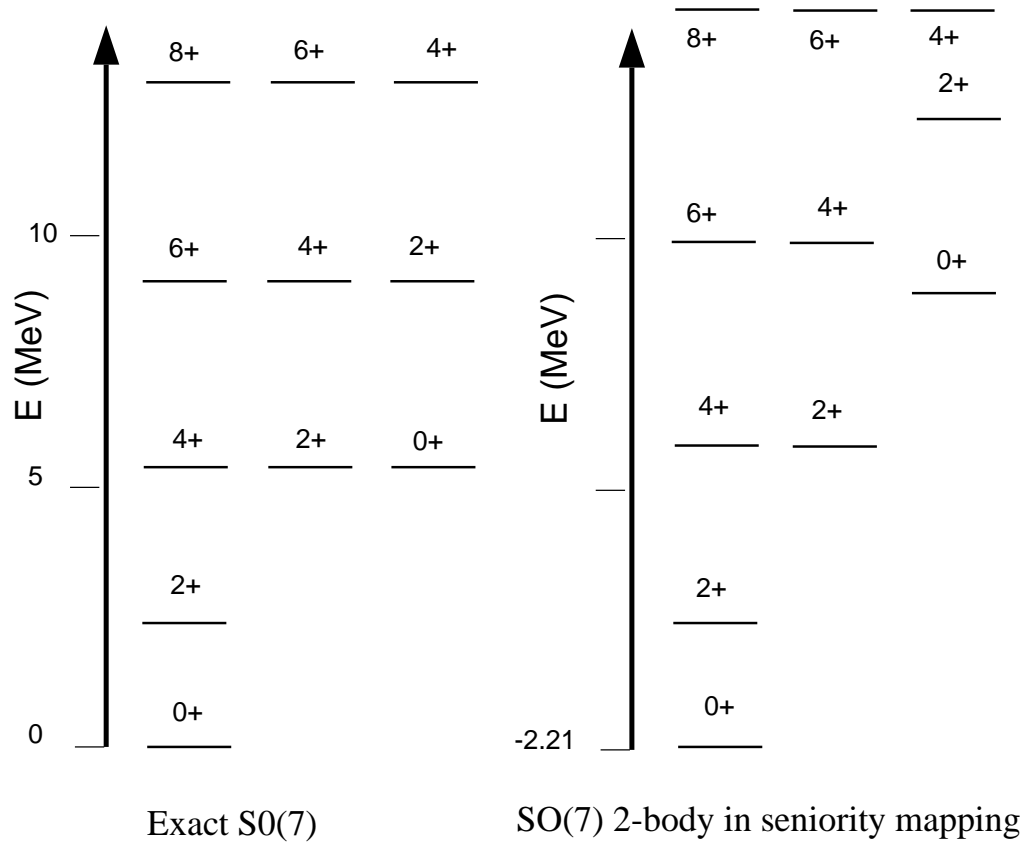


Figure 1